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Flow development in the vicinity of the sharp trailing edge on bodies impulsively set into motion. Part 2

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Recently, Williams (1982) carried out a study of the initial development of the viscous flow in the vicinity of a sharp trailing edge on a symmetrical body impulsively set into motion. The numerical results of that study indicate that, for small or moderate trailing-edge angles, a moving singularity occurs in the solution fairly early in the flow development and that the flow in the vicinity of this singularity exhibits the characteristics of unsteady separation. In the present study, this problem is re-examined with the objective of providing convincing evidence for the existence of such a singularity and describing its structure.

A detailed asymptotic theory is developed for the structure of the boundary-layer solution in the vicinity of the moving singularity. The major features of this theory are then tested by comparison with careful numerical solutions carried as closely as possible to the singularity. The agreement between the asymptotic theory and the numerical integration of a the boundary-layer equations is favourable, and it is concluded that the proposed structure of the singularity is correct for unsteady flow past a sharp trailing edge that is impulsively set into motion.

1. Introduction

When a symmetrical body is impulsively set into motion, at time $t = 0$, with a uniform motion along the plane of symmetry, the inviscid flow over the body develops instantaneously, while, in contrast, the flow within the viscous layer adjacent to the body develops slowly. The fully developed steady-state viscous flow is reached only after some period of time. The development of the viscous layer actually occurs in two stages. For small times, the flow develops locally under the influence of local forces and acceleration and is largely independent of any upstream influence. At some later time, the influence of the body leading edge comes more strongly into play and the flow develops under the influence of both local conditions and the conditions far upstream (at the leading edge).

In a recent paper, Williams (1982; hereinafter referred to as I) investigated the initial development of a viscous flow in the vicinity of the sharp trailing edge on a symmetrical body which is impulsively set into motion. For this problem, the boundary-layer equations may be reduced to a semisimilar form in terms of the parameter m, where the internal angle of the trailing edge $M\pi$ is related to m by

$$M\pi = \frac{2\pi m}{1+m}.$$

Thus $m = 0$ corresponds to a cusped trailing edge and $m = 1$ to the rear stagnation point on a bluff body. It has been shown by Proudman & Johnson (1962) (see also Robins & Howarth 1972) that if $m = 1$ a solution exists for all time and at large times the solution is double-structured with an outer part scaling exponentially with time. Williams found that for all chosen values of m in the range $0 < m < 1$ the solution of the reduced equations terminated owing to the occurrence of a singularity. In the physical plane this singularity moves up the body from the trailing edge with a finite velocity. It was shown that, in a coordinate system moving with the singularity, as the flow approaches the singularity the velocity profile approaches one in which the shear and velocity vanish simultaneously at a point within the boundary layer. These results substantiate the Moore–Rott–Sears model for unsteady separation, and the physical picture that emerges is one in which the unsteady separation point originates at the trailing edge and moves forward along the body as time proceeds. For $m = 0$ there are objections on physical grounds to the formulation, and the numerical integration was arbitrarily terminated before any singularity appeared.

The above results, together with other recent results, provide convincing numerical evidence that unsteady boundary-layer calculations in which there is an adverse pressure gradient are terminated by a singularity, and that the features of the flow in the vicinity of this singularity are just those associated with the well-known Moore–Rott–Sears condition. It appears therefore that there is a strong similarity between the two-dimensional steady boundary-layer calculation and two-dimensional unsteady boundary-layer calculations for flows leading to separation: in each case the calculation is terminated by a singularity. In the steady case, the singularity (the 'Goldstein singularity') is well known and well understood. The singularity occurs at the wall, and the local component of velocity parallel to the wall, for example, varies as the square root of the distance from the separation point. Unfortunately, the singularity associated with unsteady separation is not as well understood as the Goldstein singularity.

Several years ago Sychev (1979) (see also Smith 1982) proposed a new kind of singularity in boundary-layer theory for describing a breakdown of the solution centred at interior points of the flow field. The conditions for the appearance of this singularity are identical with the Moore–Rott–Sears condition; namely, that relative to axes moving with the singularity the velocity u of the fluid parallel to the wall should satisfy

$$u = \frac{\partial u}{\partial y} = 0,$$

y measuring distance from the wall. If the singularity is centred on the wall itself, then the most commonly occurring form is that described by Goldstein, and its structure is controlled by a balance between viscous and inertia forces. The Sychev singularity is by contrast entirely inviscid in origin, the viscous forces being negligible at the centre of the singularity. Although this singularity was described in the context of steady flow, it may also be relevant to unsteady problems of the semisimilar type and even more generally (van Dommelen & Shen 1981). The aim of the present paper is to show that the singularity is relevant to the calculation in I. The specific comparison is made for $m = 0.2$ and the relevance is inferred for $0 < m < 1$. A detailed asymptotic theory is developed for the structure of the solution in the vicinity of moving singularity. To test this theory, new numerical solutions to the boundary-layer equations are obtained for this problem. These solutions are carried as closely as possible to the singularity and are designed to provide detailed information for comparison with the theory. The agreement between the asymptotic theory and the numerical integration of the boundary-layer equations is favourable.

2. The basic equations

The unsteady boundary-layer equations are (Rosenhead 1963):

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}+\frac{\partial^2 u}{\partial y^2}, \tag{2.1a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1b}$$

where, after suitable scaling, x and y denote distances along and normal to the wall, u and v are the corresponding velocity components, t is time, and $u_e(x,t)$ is the given external velocity. We denote the origin of the coordinate system by O and we take O to be at the vertex of the trailing edge. The boundary conditions, associated with these equations and corresponding to an impulsive start at $t=0$, are

$$u=v=0 \quad (y=0,\ x\leqslant 0,\ t\geqslant 0), \tag{2.2a}$$

$$u\to u_e(x,t) \quad (y\to\infty,\ x<0,\ t>0), \tag{2.2b}$$

$$u=u_e(x,0) \quad (y>0,\ x<0,\ t=0), \tag{2.2c}$$

$$u=u_e(0,t) \quad (y>0,\ x=0,\ t>0). \tag{2.2d}$$

Further, we take

$$u_e=+(-x)^m \quad (0<m\leqslant 1,\ t>0), \tag{2.3}$$

and simplify the equations by writing

$$\tau=(-x)^{m-1}t, \quad \eta=y(-x)^{\frac{1}{2}(m-1)}, \quad \psi=(-x)^{\frac{1}{2}(1+m)}f(\eta,\tau), \tag{2.4}$$

where $\partial\psi/\partial y=u$, $\partial\psi/\partial x=-v$. It then follows that

$$f_{\eta\eta\eta}-f_{\eta\eta}[\tfrac{1}{2}(1+m)f-(1-m)\tau f_\tau]-f_{\eta\tau}[1+(1-m)\tau f_\eta]=m(1-f_\eta^2), \tag{2.5}$$

where subscripts denote differentiation with respect to either η or τ, and the corresponding boundary conditions are

$$f(0,\tau)=f_\eta(0,\tau)=0, \quad f_\eta(\infty,\tau)=1 \quad (\tau>0), \tag{2.6a}$$

$$f(\eta,0)=1 \quad (\eta>0). \tag{2.6b}$$

The underlying assumption here is that, although the boundary layer on the body terminates at O, there is a sense in which the structure of the boundary layer near O is fixed by local conditions and is independent of previous history.

As noted in I for any flow that is impulsively set into motion, the viscous layer next to the body develops in two phases. In the first of these, the local flow develops largely independently of the upstream conditions. That is, it takes a finite amount of time, from the onset of the motion, for the effect of a sharp leading edge to be felt at points downstream. During this period the flow develops under the influence of local conditions alone. At some later time, after the effect of the leading edge has been felt at the point in question, the local flow develops under the influence of the leading edge as well as the local conditions. The present calculations are related only to that initial phase of the motion in which the flow develops under the influence of local conditions.

In §3, a detailed asymptotic theory is developed for the structure of the solution of (2.5) in the vicinity of the moving singularity. To test this theory, it is desirable to have a numerical solution of (2.5) in the vicinity of the singularity. Unfortunately, it is not possible to directly solve (2.5) numerically because the solution at $\tau = 0$ is not readily obtainable. To avoid this problem, we introduce the transformation

$$\eta = (1-e^{-\tau})^{\frac{1}{2}} N, \quad f(\tau,\eta) = (1-e^{-\tau})^{\frac{1}{2}} F(\tau,\eta).$$

In terms of F and N, (2.5) becomes

$$\frac{\partial^3 F}{\partial N^3} - \frac{\partial^2 F}{\partial N^2}\left\{F[\tfrac{1}{2}(1+m)(1-e^{-\tau}) - \tfrac{1}{2}(1-m)\tau e^{-\tau}] - (1-m)(1-e^{-\tau})\tau\frac{\partial F}{\partial \tau}\right\}$$
$$+ e^{-\tau}\frac{N}{2}\frac{\partial^2 F}{\partial N^2} - \frac{\partial F^2}{\partial \tau\,\partial N}(1-e^{-\tau})\left\{1+(1-m)\tau\frac{\partial F}{\partial N}\right\}$$
$$= (1-e^{-\tau})\,m\left\{1-\left(\frac{\partial F}{\partial N}\right)^2\right\}. \quad (2.7)$$

This transformation was made so that the solution to (2.7) remains finite as $\tau \to 0$. In fact, at $\tau = 0$, the solution to (2.7) is a similar solution, and thus the numerical solution is easily started. Furthermore, it is clear that for large τ (near the singularity) (2.7) and (2.5) have the same behaviour. A careful and detailed solution of (2.7) was obtained to test the results of the asymptotic solution.

The numerical method employed in solving (2.7) is exactly the same as that employed in I. Equation (2.7) is written as a pair of equations:

$$\frac{\partial F}{\partial N} = w,$$

$$\frac{\partial^2 w}{\partial N^2} + \alpha_1\frac{\partial w}{\partial N} + \alpha_2 w + \alpha_3 = \alpha_4\frac{\partial w}{\partial \tau},$$

in which

$$\alpha_1 = -F[\tfrac{1}{2}(1+m)(1-e^{-\tau}) - \tfrac{1}{2}(1-m)\tau\, e^{-\tau}] + (1-m)(1-e^{-\tau})\tau\frac{\partial F}{\partial \tau} + \tfrac{1}{2}e^{\tau}N,$$

$$\alpha_2 = m(1-e^{-\tau})\frac{\partial F}{\partial N}, \quad \alpha_3 = -m(1-e^{-\tau}), \quad \alpha_4 = (1-e^{-\tau})\left\{1+(1-m)\tau\frac{\partial F}{\partial N}\right\}.$$

In this system the boundary conditions are

$$w(\tau,0) = F(\tau,0) = 0, \quad \lim_{N\to\infty} w(\tau,N) = 1.$$

The differentials in the N-direction are represented by central differences, while those in the τ-direction are represented by backwards differences. At the first station a similar solution exists, so that no backwards differences are required. At the second station a two-point-backwards difference representation is used for the τ-derivatives; at the third and subsequent stations, three-point-backwards differences are used. The solution is begun at $\tau = 0$ and marched in the direction of increasing τ. At each station the set of equations for the values of $w(\tau, N)$ at each grid point is a tradiagonal system, which is solved by employing the Thomas algorithm. Since the coefficients in the tridiagonal system contain the unknown functions $F(\tau, N)$ and $\partial F(\tau, N)/\partial N$, the system is solved in an iterative fashion. An initial guess is made for the discrete values of $w(\tau, N)$ at each grid point at a given τ-station, and this guess is used to calculate the coefficients α_1, α_2, α_3 and α_4. The tridiagonal system is then solved for a new approximation to $w(\tau, N)$. At the end of each iteration, new coefficients α_1, α_2, α_3 and α_4 are calculated employing the current solution for $w(\tau, N)$. This procedure is repeated until the values of $w(\tau, N)$ at each grid point for two successive iterations

τ	C_f	δ^*	X	Z	η_0
0	∞	0	—	—	—
1	0.4034	1.6389	—	—	—
2	0.1689	2.2812	—	—	—
3	0.0382	3.0845	—	—	—
4	−0.0617	4.1381	0.3027	0.1665	0.339
5	−0.1577	5.848	0.1773	0.1208	1.009
5.2	−0.1796	6.466	0.1421	0.1067	1.227
5.4	−0.2045	7.541	0.0925	0.0830	1.599
5.45	−0.2115	8.056	0.0742	0.0721	1.779
5.50	−0.2190	9.041	0.0476	0.0528	2.132
5.51	−0.2205	9.432	0.0396	0.0461	2.276
5.52	−0.2221	10.068	0.0293	0.0364	2.514
5.53	−0.2238	12.316	0.0099	0.0141	3.375

TABLE 1

agree to within some small tolerance. This scheme is similar to that proposed by Blottner (1972). The grid spacing in the N-direction was 0.1, while that in the τ-direction was initially 0.02 but was gradually decreased to 0.005, 0.001 and finally to 0.0005 as the singularity is approached. This decrease in the τ-step size was chosen to provide high resolution of the solution in the vicinity of the singularity.

The solutions indicate that $\partial f/\partial \eta = \partial F/\partial N$ is positive for all $\eta(N)$ for small τ. At some value of τ the velocity profile $w(\tau, N)$ begins to show reversed flow near $\eta(N) = 0$ while still remaining positive for large $\eta(N)$. With a further increase in τ the region of reversed flow increases rapidly and the solution terminates in the vicinity of $\tau = \tau_s$, where

$$1+\tau(1-m)\frac{\partial F}{\partial N} = 0; \tag{2.8}$$

i.e. when the coefficient of $\partial^2 F/\partial N\,\partial\tau$ in (2.7) (or $\partial^2 f/\partial\eta\,\partial\tau$ in (2.5)) changes sign. The physical interpretation of this result is that, for $\tau > \tau_s$, disturbances may propagate in the direction of *decreasing* τ. For the particular case of $m = 0.2$, the reverse flow starts at $\tau \approx 3.363$ and $\tau_s = 5.5312$. Some of the principal properties of the numerical solution are displayed in table 1, where we have defined

$$C_f(\tau) = f_{\eta\eta}(0,\tau), \tag{2.9a}$$

$$\delta^*(\tau) = \lim_{\eta\to\infty}(\eta - f), \tag{2.9b}$$

$$X = \left[f_\eta + \frac{1}{\tau(1-m)}\right]_{\eta=\eta_0}, \tag{2.9c}$$

$$Z = f_{\eta\eta\eta}\Big|_{\eta=\eta_0}, \tag{2.10}$$

and $\eta_0(\tau)$ is the value of η where f_η achieves its minimum value.

3. Asymptotic theory

The principal features of the structure of the solution of (2.5) as $\tau \to \tau_s$ are as follows. First there is a reversed-flow region near $\eta = 0$ and the skin friction at the wall appears to be smooth. Thus the derivative of C_f is approximately -0.1 at $\tau = 3$ and has only reached approximately -0.15 at $\tau = 5.51$. Secondly, X, evaluated at the minimum value of f_η, is decreasing precipitously towards zero as $\tau \to \tau_s$ and Z is behaving in a similar fashion. Thirdly, the displacement thickness δ^* is increasing quite rapidly and has reached an unusually large value at $\tau = 5.53$ for a smooth solution of the boundary-layer equations.

Sychev's conjecture about the nature of the singularity, which we wish to generalize and to test against this numerical data, follows from the assumptions

$$\text{(i)} \quad \eta_0 \to \infty \quad \text{as} \quad \tau \to \tau_s,$$

$$\text{(ii)} \quad F_s'(\eta) + \frac{1}{(1-m)\tau_s} \approx A\,\mathrm{e}^{-\beta\eta} \quad \text{as} \quad \eta \to \infty, \tag{3.1a}$$

where

$$F_s'(\eta) = \lim_{\tau\to\tau_s} f_\eta(\eta,\tau), \tag{3.1b}$$

and A, β are constants. The generalization that we shall also consider here is

$$\text{(iii)} \quad F_s'(\eta) + \frac{1}{(1-m)\tau_s} \approx A\eta^{-\alpha} \quad \text{as} \quad \eta \to \infty, \tag{3.1c}$$

where $\alpha > 0$ is a constant.

In physical terms, the proposal is that, as $\tau \to \tau_s$, the forward-moving fluid in the boundary layer is pushed out to an infinite distance from the wall. In figure 1 we display profiles of f_η in the range $0 < \eta < 6$, which shows this tendency, but we agree that, if correct, the mechanism is weak except when $\tau_s - \tau$ is very small.

It is convenient to divide the structure of the boundary layer near $T = 0$, where

$$T = \tau_s - \tau, \tag{3.2}$$

into three parts, each corresponding to a different range of values of η. The first, adjacent to the wall $\eta = 0$, is defined by the requirement that η be finite, and here we set

$$f = F_s(\eta) + TF_1(\eta) + \dots; \tag{3.3}$$

i.e. we assume that f may be expanded in a power series in T whose coefficients are functions of η. The validity of this hypothesis is verified by the numerical results presented in figure 1. The leading term $F_s(\eta)$ is arbitrary except that the usual conditions at $\eta = 0$ and the conjectures (3.1) must hold. On substituting (3.3) into (2.5), we find immediately that

$$F_1(\eta) = -[1+\tau_s(1-m)F_s'] \int_0^\eta \frac{F_s''' + m(F_s'^2 - 1) - \frac{1}{2}(m+1)F_sF_s''}{[1+\tau_s(1-m)F_s']^2}\,d\eta. \tag{3.4}$$

The boundary conditions on F_1 at $\eta = 0$ are automatically satisfied in view of the properties of F_s. We insist that $F'(0) = 0$ because it has not proved possible to smooth out a discontinuity of F_1' at $\eta = 0$ by means of a sub-boundary layer. The expansion (3.3) is well ordered except as $\eta \to \infty$ by virtue of (3.1). For then we see that

$$F_1(\eta) \sim \frac{m[\{\tau_s(1-m)\}^2 - 1]}{2A[\tau_s(1-m)]^3\beta}\,\mathrm{e}^{\beta\eta} \tag{3.5a}$$

if (ii) holds, and

$$F_1(\eta) \approx \frac{m[\{\tau_s(1-m)\}^2 - 1]}{A[\tau_s(1-m)]^3(2\alpha+1)}\,\eta^{\alpha+1} \tag{3.5b}$$

if (iii) holds. We shall pursue the consequences of (3.5a) here and defer the discussion of conjecture (iii) to the appendix.

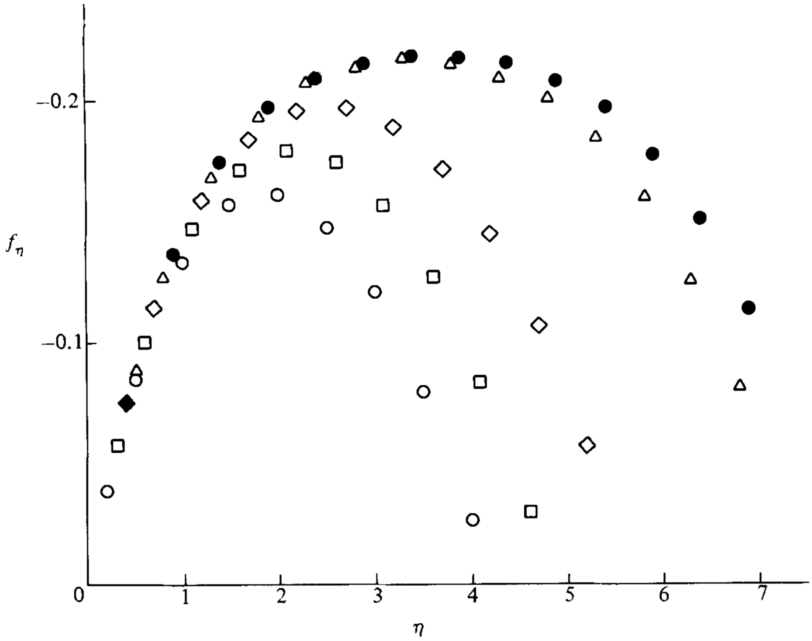

FIGURE 1. Variation of f_η with η near the wall ($\eta = 0$): ○, $\tau = 4.45$; □, 5.50; ◇, 5.52; △, 5.53; ●, 5.5305.

It follows from (3.5*a*) that the first two terms of the series (3.3) are in balance when

$$e^{2\beta\eta} \approx T^{-1}; \quad \text{i.e.} \quad \eta \approx \frac{1}{2\beta}\log T^{-1}, \tag{3.6}$$

and this suggests that we match the above solution to that in a second layer above it defined by

$$\eta = \eta_s(T) + \xi, \quad \text{with} \quad \eta_s \sim -\frac{1}{2\beta}\log TD^2 \quad \text{as} \quad T \to 0, \tag{3.7}$$

where $\xi = O(1)$ and D is a constant to be specified below. Thus η_s has a logarithmic singularity in T as $T \to 0$, in agreement with (3.6), and later on we will identify η_s with η_0. When $\xi \approx 1$, $X = O(T^{\frac{1}{2}})$, and this suggests that we write

$$f = -\frac{\eta_s(T)+f_0+\xi}{\tau_s(1-m)} + \frac{(1+m)T}{2\tau_s^2(1-m)^2}(\eta_s+f_0+\xi) + T^{\frac{1}{2}}g(\xi, T), \tag{3.8}$$

where f_0 is a constant. Then g satisfies

$$\begin{aligned}\tfrac{1}{2}\tau_s(1-m)\,[g_\xi^2 - g g_{\xi\xi}] - m[1-\{\tau_s(1-m)\}^{-2}]\\ = -T^{\frac{1}{2}}g_{\xi\xi\xi} - \left[\frac{(1+m)T^{\frac{1}{2}}}{2\tau_s^2(1-m)} + Tg_{T\xi}\right]\left[\frac{3-m}{2\tau_s(1-m)}T^{\frac{1}{2}} + (1-m)\tau_s g_\xi\right]\\ + \tau_s(1-m)\,Tg_T g_{\xi\xi} - \tfrac{1}{4}T^{\frac{1}{2}}\frac{3-m}{\tau_s(1-m)}g_\xi + \eta_s T^{\frac{1}{2}}\frac{3-m}{2\tau_s(1-m)}g_{\xi\xi}\\ + \frac{2mT^{\frac{1}{2}}}{\tau_s(1-m)}g_\xi + O(T).\end{aligned} \tag{3.9}$$

The natural expansion of g is in powers of $T^{\frac{1}{2}}$, and so we write

$$g = g_0 + T^{\frac{1}{2}}g_1 + \ldots. \tag{3.10}$$

The equation satisfied by g_0 is

$$g_0'^2 - g_0 g_0'' = \lambda^2, \tag{3.11}$$

where

$$\lambda^2 = \frac{2m[\tau_s^2(1-m)^2-1]}{[\tau_s(1-m)]^3} \tag{3.12}$$

and λ is real. The solutions of (3.11) are either trigonometrical or hyperbolic functions, of which only the second kind can match with (3.3). Hence

$$g_0 = \lambda\beta^{-1}\sinh\beta\xi, \tag{3.13}$$

and the matching now follows. It is noted that, if the imposed pressure gradient is favourable, $\lambda^2 < 0$ and the appropriate solution of (3.11) is $\beta^{-1}(-\lambda^2)^{\frac{1}{2}}\cosh\beta\xi$, which corresponds to a point of inflexion of the profile and to X changing sign as ξ passes through zero (Elliot, Cowley & Smith 1983). We shall refer to this in the appendix. The principal property of η_s is confirmed, and on continuing the expansion we find that

$$g_1 = -\frac{1}{3\beta\tau_s(1-m)}\left[\frac{5-7m}{4\tau_s(1-m)}+\beta^2\right][3+e^{2\beta\xi}]+\frac{1}{4\beta}\frac{3-m}{\tau_s^2(1-m)^2} + A_1\cosh\beta\xi + B_1(3+\cosh 2\beta\xi), \tag{3.14}$$

where A_1 and B_1 are arbitrary constants. As $\xi \to -\infty$, it follows that

$$f_\eta \sim -\frac{1}{\tau_s(1-m)} + T^{\frac{1}{2}}[\tfrac{1}{2}\lambda\,e^{\beta\xi}+\tfrac{1}{2}\lambda\,e^{-\beta\xi}] - \tfrac{1}{2}\beta A_1 T e^{-\beta\xi} - \beta B_1 T e^{-2\beta\xi} + \frac{1}{2}\frac{(1+m)T}{\tau_s^2(1-m)^2} + O(T^{\frac{3}{2}}). \tag{3.15}$$

If we replace ξ by η, using (3.7), and set $D = \lambda/2A$

$$f_n = -\frac{1}{\tau_s(1-m)} + A\,e^{-\beta\eta} - 4\beta B_1 A^2\lambda^{-2}e^{-2\beta\eta} - 2\beta A A_1\lambda^{-1}T^{\frac{1}{2}}e^{-\beta\eta} + \frac{\lambda^2 \Gamma}{4A}e^{\beta\eta} + \dots \quad (\eta \gg 1). \tag{3.16}$$

It is not possible to match the terms depending on A_1 in (3.16) to the solution in $\eta = O(1)$, and so $A_1 = 0$. The other terms either match with (3.3) or in the case of $-\beta B_1 e^{-2\beta\eta}$ is allowable in the properties of F_s.

As $\xi \to \infty$

$$f \sim -\frac{\xi+\eta_s}{\tau_s(1-m)} + \frac{\lambda}{2\beta}T^{\frac{1}{2}}e^{\beta\xi} + CT\,e^{2\beta\xi} + O(\xi T, T^{\frac{3}{2}}), \tag{3.17}$$

where C is a constant, and this form suggests that in the outer region we write

$$Y = \xi - \frac{1}{2\beta}\log T^{-1} = \eta + \frac{1}{\beta}\log DT. \tag{3.18}$$

Then when Y is large and negative

$$f \sim -\frac{2\eta_s(T)}{\tau_s(1-m)} - \frac{Y}{\tau_s(1-m)} + \frac{\lambda}{2\beta}e^{\beta Y} + C\,e^{2\beta Y} + \dots, \tag{3.19}$$

and in turn this suggests that when $Y = O(1)$

$$f + \frac{2\eta_s(T)}{\tau_s(1-m)} \to \hat{f}_s(Y) \quad \text{as} \quad T \to 0, \tag{3.20}$$

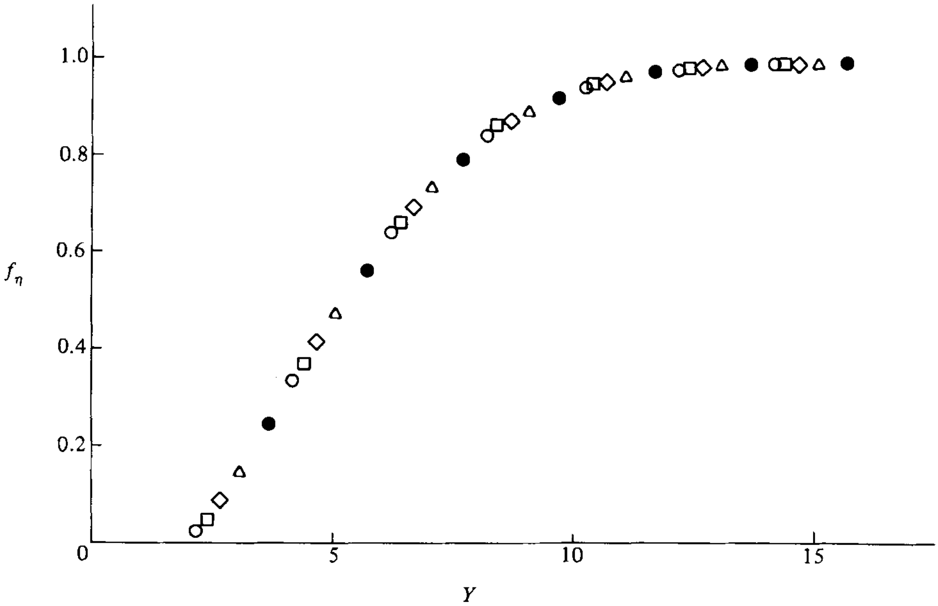

FIGURE 2. Variation of f_η with the coordinate Y in the outer portion of the layer: ○, $\tau = 5.45$; □, 5.50; ◇, 5.52; △, 5.53; ●, 5.5305.

where $\hat{f}$ is independent of T. As $Y \to -\infty$

$$\hat{f} \sim -\frac{Y}{\tau_s(1-m)} + \frac{\lambda}{2\beta} e^{\beta Y} + \dots, \tag{3.21}$$

and it seems reasonable to expect that

$$\hat{f}_Y \sim -\frac{1}{\tau_s(1-m)} + \frac{\lambda}{2} e^{\beta Y} \quad \text{as} \quad Y \to \infty, \tag{3.22}$$

to be compatible with the boundary conditions of (2.6). This conclusion is tested in figure 2, where plots of f_η against Y are drawn for various values of T, the value of β being chosen, in anticipation of (4.6), to be 1.23. The collapse of the curves as $T \to 0$ is encouraging. It should be noted that the expansion (3.10) is not entirely straightforward since the ordinary differential equations that determine g_n have regular singularities when $g_0 = 0$.

4. Comparisons

In this section we shall compare the asymptotic theory of the singularity at $T = 0$ in detail with significant properties of the numerical solution. The first comparison is made for X (2.9*c*). So long as only the first two terms of (3.10) are taken into account, this quantity may be computed at ζ = where $g_{0\zeta\zeta} = 0$, and is

$$X = T^{\frac{1}{4}}\left[\lambda - \frac{2T^{\frac{1}{4}}}{3\tau_s(1-m)}\left\{\beta^2 - \frac{1+m}{\tau_s(1-m)}\right\}\right] + \dots. \tag{4.1}$$

A check on the theory is therefore provided by tabulating X^2 as a function of τ, which is done in table 2. In addition, X^2, obtained from the numerical calculation, is plotted as a function of τ, for the region very near the singularity, in figure 3. The linear behaviour of X^2 as $X \to 0$ is largely confirmed, and we estimate that $\tau_s = 5.5312$ so that $\lambda = 0.2929$.

τ	10^4X^2	Δ^*	Δ_0	$Z/X\beta^2$	T^*
4	916	4.564	0.512	0.364	2.477
5	314	5.216	0.752	0.450	0.650
5.2	202.0	5.361	0.778	0.496	0.381
5.4	85.59	5.510	0.773	0.593	0.140
5.45	55.03	5.545	0.758	0.642	0.0849
5.50	22.62	5.574	0.723	0.733	0.0318
5.51	15.72	5.578	0.709	0.769	0.0215
5.52	8.57	5.576	0.688	0.821	0.0113
5.53	0.98	5.591	0.641	0.942	0.0012

TABLE 2

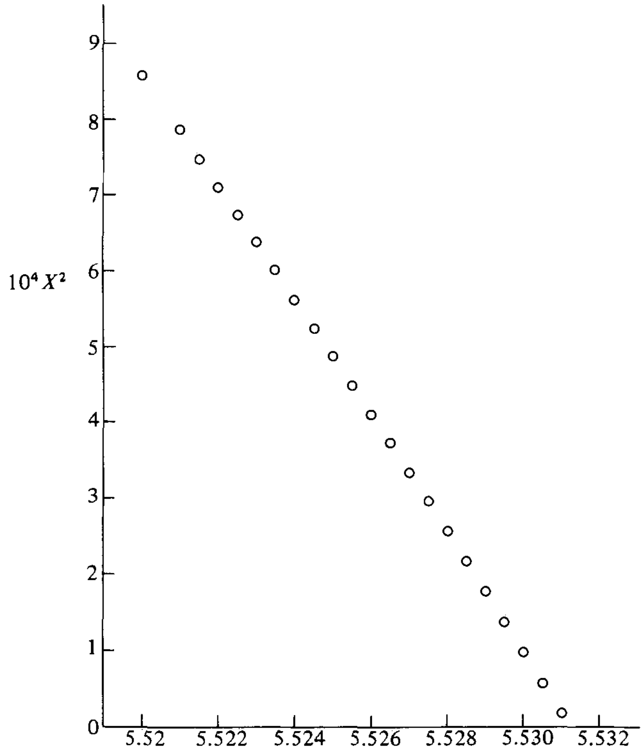

FIGURE 3. Variation of X with τ near $T = 0$.

Next we examine the behaviour of

$$\delta^* = \int_0^\infty (1-f_\eta)\,\mathrm{d}\eta = \lim_{\eta\to\infty} (\eta - f). \tag{4.2}$$

If $\eta = Y + 2\eta_s$ and $Y = O(1)$, $T \ll 1$, the asymptotic theory suggests that

$$f = \frac{2\eta_s}{\tau_s(1-m)} + \hat{f}(Y) + O(1), \tag{4.3}$$

where $\hat{f}_Y \to 1$ as $Y \to \infty$. Hence

$$\delta^* = \frac{1}{\beta}\frac{\tau_s(1-m)+1}{\tau_s(1-m)} \log T^{-1} + O(1) \tag{4.4}$$

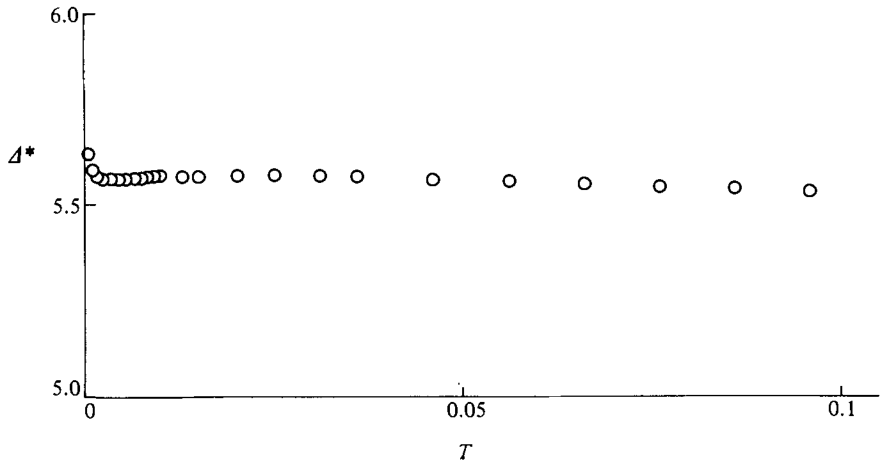

FIGURE 4. Variation of Δ^* with T near $T = 0$.

as $T \to 0$. We tabulate

$$\Delta^* = \delta^* + \log T \tag{4.5}$$

in table 2 and plot this quantity as a function of T, near the singularity, in figure 4. It is seen that Δ^* is almost constant as $T \to 0$, suggesting that

$$\beta \approx 1.23, \tag{4.6}$$

although errors of about 5 % can be expected in view of the behaviour of $\log T$ when $T \ll 1$. We note that $\Delta^*(T)$ deviates from a constant value slightly at $T = 0.0012$ and sharply at $T = 0.0007$. This deviation appears to be the direct result of the limitation of the calculations to a finite region in the η-direction. As the singularity is approached and the thickness of the boundary layer grows rapidly, the limited computational region inhibits the natural asymptotic behaviour in the outer regions of the boundary layer. As a result, the physical characteristics of the boundary layer which depend to a large extent on the behaviour of the outer reaches of the boundary layer (e.g. δ^*) are somewhat in error very near the singularity.

We may now test the asymptotic prediction of η_0 by tabulating

$$\Delta_0 = \eta_0 + \frac{1}{2\beta} \log T, \tag{4.7}$$

and this is also displayed in table 2, and, for small T, in figure 5. It is seen that Δ_0 is not quite constant near $T = 0$, but the variation is perhaps acceptable in view of the relative smallness of η_0. The behaviour of Z (2.9) near $T = 0$ may also be found from (3.10), and we have

$$Z = \lambda\beta^2 T^{\frac{1}{2}} - \frac{8\beta^2 T}{3\tau_s(1-m)} \left\{ \frac{5-7m}{4\tau_s(1-m)} + \beta^2 \right\} + \dots, \tag{4.8}$$

and so

$$\frac{Z}{X\beta^2} = 1 - 2.9T^{\frac{1}{2}} + \dots \tag{4.9}$$

when $T \ll 1$, taking $\tau_s = 5.5312$ and $\beta = 1.23$. The function $Z/X\beta^2$, calculated from the numerical results, is tabulated in table 2 and we see that it is moderately consistent with (4.9), but the differences are significant. For example, when $T = 0.0012$, the predicted value is 0.900, whereas the computed value is 0.942.

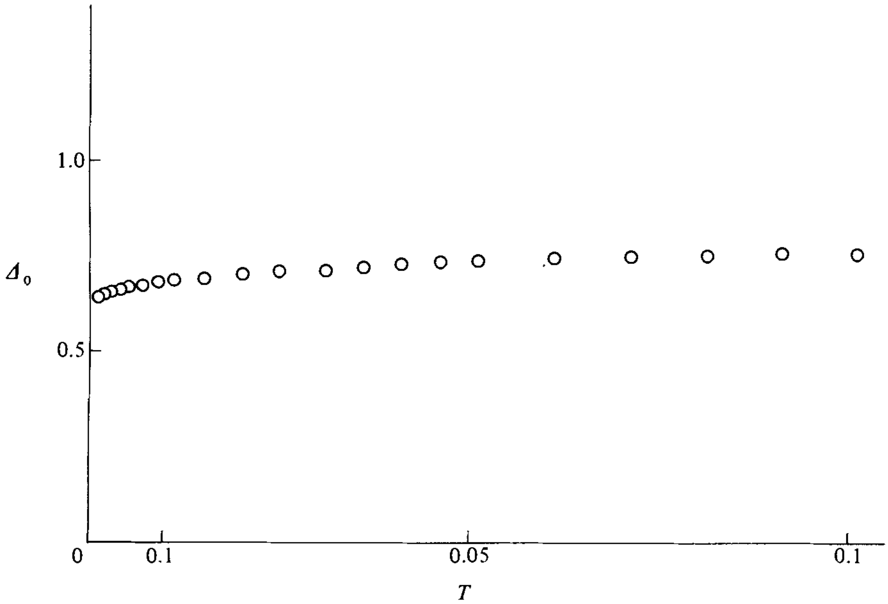

FIGURE 5. Variation of Δ_0 with T near $T = 0$.

However, it only needs a small change in β to effect a considerable improvement. If $\beta = 1.25$, which is within the allowable range of β already mentioned, this computed value changes to 0.913. For such a value of β, Δ^* has a greater variation with T near $T = 0$ than if $\beta = 1.23$, but it is almost linear and might be accounted for by including higher powers of T in the expansion of $g(\beta, T)$. Also the corresponding changes in Δ_0 are insignificant. Be that as it may, it should also be borne in mind that the numerical values of the coefficients in (4.8) suggest that this asymptotic formula for Z has a small domain of validity. A more useful comparison can be carried out with X. We retain the value 1.23 for β, and then (4.1) implies that

$$X = 0.2929T^{\frac{1}{2}} - 0.187T + O(T^{\frac{3}{2}}) \tag{4.10}$$

and we test its validity by considering the inverse function

$$T^* = 11.66X^2 + 50.8X^3,$$

which should be equal to $T + O(T^2)$. A set of values of T^* is given in table 2 and we see that this prediction is confirmed to three significant figures.

Finally, an additional visual check is provided in figure 6 by plotting

$$\left[f_\eta + \frac{1}{\tau(1-m)}\right] T^{-\frac{1}{2}}$$

as a function of $\eta - \eta_0$ for various values of τ as well as the limit solution $g_0'(\xi)$. The collapse of the data on to the limit curve is satisfactory.

In conclusion, we claim that the agreement between the asymptotic expansion and the numerical integration of the equation is favourable and that Sychev's conjecture about the structure of the singularity is correct for unsteady flow past a trailing edge for $m = 0.2$. The analytic argument holds for all m in $0 < m < 1$, and the existence of a singularity is strongly indicated by the numerical calculation in I for other values of m in this range. We infer therefore that for all such m, i.e. trailing edges with internal angles that are either acute or obtuse, the solution of the equation (2.5) terminates in a singularity with the Sychev structure.

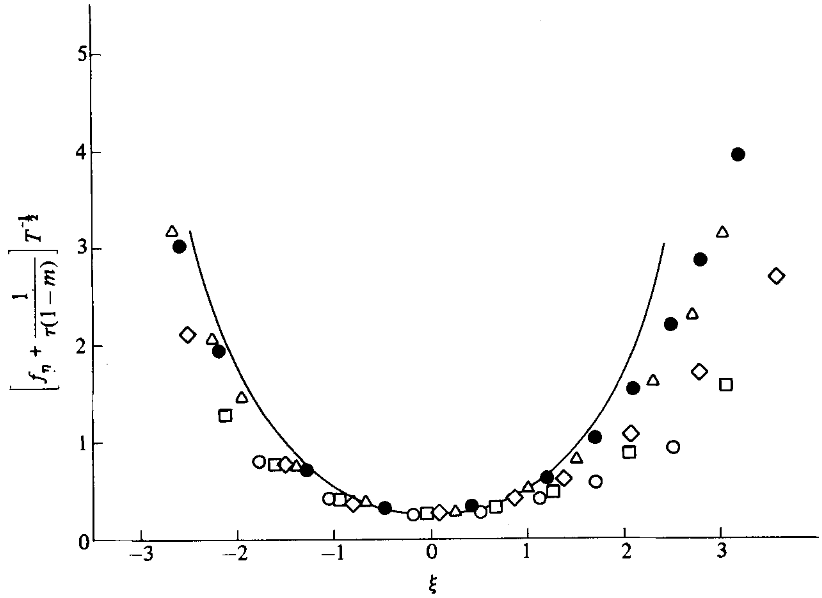

FIGURE 6. Variation of $\{f + 1/\tau(1-m)\}\,T^{-\frac{1}{2}}$ with coordinate ξ: ○, $\tau = 5.45$; □, 5.50; ◇, 5.52; △, 5.53; ●, 5.5305; ——, first term of the series (from (3.13)).

5. Discussion

Semisimilar flows are the simplest class of unsteady boundary layers to which the MRS condition for a singularity might apply. In this class, the dependence of the solution on x, t may be reduced to a single variable τ as in (2.4), and the governing equation takes the form

$$f_{\eta\eta\eta} - \theta_1 f_{\eta\tau} - \theta_2 f_{\eta\eta} = \theta_3, \tag{5.1}$$

where θ_1, θ_2, θ_3 are functions of f, f_τ, f_η, η, τ only. The singularity occurs when $\theta_1 = f_{\eta\eta} = 0$. Let this point be designated $\tau = \tau_s$, $\eta = \eta_s$.

Most commonly $\theta_1 \geqslant 0$ for $\tau < \tau_s$, and, when $\tau > \tau_s$, there is a range of values for η for which $\theta_1 < 0$. If $\eta_s = 0$, the singularity is probably of the Goldstein (1948) type, viscosity playing a significant role in its structure, but an exception has recently been found, in a more complicated situation, by Simpson & Stewartson (1982). If $\theta_3 = 0$, the solution may be characterized by an essential singularity, so that it remains smooth in all its derivatives as τ increases through τ_s. An example is provided by the impulsive motion of a flat plate (Stewartson 1951, 1973) and in this solution $\eta_s = \infty$.

The structure proposed by Sychev and generalized in the appendix is also relevant to flows for which $\theta_1 \geqslant 0$ for $\tau < \tau_s$ and at the singularity $\eta_s = \infty$ and $\theta_3 > 0$; i.e. the pressure gradient is adverse. Of the possible forms of the singularity, Sychev's is the weakest, with only a logarithmic singularity in the displacement thickness, and seems to be the most likely to occur. Previously, computations exhibiting an apparently singular form when θ_1 first changes sign have been carried out by Telionis & Werle (1973) for the steady boundary layer on a downstream moving wall and by Williams & Johnson (1974) for an external velocity

$$u_e = 1 - \frac{Ax}{1-Bt}, \tag{5.2}$$

where A, B are positive constants. The impression left by these studies and by Telionis' (1982) general review of unsteady boundary layers is that in both cases η_s is finite. The present investigation does, however, suggest strongly that η_s might be infinite, and it would be of interest to re-examine these solutions with the object of testing this conjecture and indeed of elucidating the structure of the singularity.

The only example known to us of a solution to which η_s is certainly finite is in the problem of an impulsively heated flat plate in air at rest (Ingham 1978; Brown & Riley 1973). Here the velocity and temperature fields are governed by a pair of equations each similar to (5.1), and to date the numerical properties of the solution near $\tau = \tau_s$ have defied analysis.

A singularity in the solution of (5.1) may be possible even when θ_1 changes sign from the outset ($\tau = 0$) of the calculation. A likely example is the steady boundary layer near an upstream-moving wall (Tsahalis 1977). In this instance, however, there is no clear indication that $f_{\eta\eta}$ and θ_1 vanish together in the singularity profile; indeed, Inoue's (1981) studies of the same problem, but at finite Reynolds number, suggest that this is not the case. From the rapid increase of δ^* as τ increases in Tsahalis' calculations, we may reasonably conjecture that $\delta^* \to \infty$ as $\tau \to \tau_s$. If we grant further that $\eta_s = \infty$, the generalized Sychev theory, described in the appendix, may be relevant. In that event θ_1 and $f_{\eta\eta}$ remain small for a large range of values of η when $\tau_s - \tau$ is small, but do not vanish together, and it is not even necessary that $f_{\eta\eta}$ vanish in this range. There is a sense in which the limit form of $f_{\eta\eta}$ has two zeros, one on either side of $\theta_1 = 0$ and, in terms of η, a large distance apart. This structure is different from that envisaged in the MRS singularity. It should be noted that the validity of the generalized-Sychev structure has not been completely established, and an alternative has been proposed by van Dommelen & Shen (1983) in which the MRS criterion does hold and δ^* appears to remain finite as $\tau \to \tau_s$. Moreover, since disturbances can travel upstream when $\theta_1 < 0$, Tsahalis, in his computation, had to approach the steady-state solution via an unsteady boundary layer. He found that a limit solution is being approached as t increases, but could not exclude the possibility that in fact a singularity is developing in the unsteady equation at a large, but finite, time. In that event, a third structure, proposed by van Dommelen (1981) for general unsteady boundary layers, might be more relevant to his problem.

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Appendix. Generalized Sychev singularity

With the assumptions (i) and (iii) of §3, we see that when $T \ll 1$ and η is large but finite

$$f_\eta \approx -\frac{1}{(1-m)\tau_s} + A\eta^{-\alpha} + \frac{mT}{A}\frac{(\tau_s(1-m))^2-1}{[\tau_s(1-m)]^3}\frac{\alpha+1}{2\alpha+1}\eta^\alpha + \dots. \tag{A 1}$$

The last two terms of (A 1) are in balance when $T \sim \eta^{-2\alpha}$, and this suggests that we introduce a new variable

$$\zeta = \eta T^{1/2\alpha} \tag{A 2}$$

so that $\zeta \to 0$ corresponds to $\eta \to \infty$. We also write

$$f = -\frac{\eta}{\tau_s(1-m)} + \eta_1 + T^{(\alpha-1)/2\alpha} g(\zeta) \tag{A 3}$$

when $\zeta \sim 1$, where η_1 is a constant, and in order to match with (A 1)

$$\frac{\partial g}{\partial \zeta} \sim A\zeta^{-\alpha} + \frac{m(\alpha+1)}{A(2\alpha+1)} \frac{[\tau_s(1-m)]^2-1}{\{\tau_s(1-m)\}^3} \zeta^{\alpha} \tag{A 4}$$

as $\zeta \to 0$. When (A 3) is substituted into the governing equation (2.5) we obtain

$$g'^2 - \frac{\alpha-1}{\alpha} g g'' = \frac{2m}{[(1-m)\,\tau_s]^3} [\{\tau_s(1-m)\}^2-1] + O(T^{\frac{1}{2}}), \tag{A 5}$$

which suggests that the error in (A 3) is $O(T^{(2\alpha-1)/2\alpha})$. The solution of this equation satisfying (A 4) is

$$g'^2 = \lambda^2 + B(-g)^{2\alpha/(\alpha-1)}, \tag{A 6}$$

where $B = A^{-2/(\alpha-1)}(\alpha-1)^{2\alpha/(\alpha-1)}$, provided that $g < 0$.

In general, it comes to an end therefore when $g = 0$, and, if ζ is then equal to ζ_0,

$$\zeta_0 = \int_{-\infty}^{0} \frac{\mathrm{d}g}{[\lambda^2 + B(-g)^{2\alpha/(\alpha-1)}]^{\frac{1}{2}}}. \tag{A 7}$$

Also, near $\zeta = \zeta_0$

$$g = \lambda(\zeta-\zeta_0) - \frac{B}{2\lambda} \frac{\alpha-1}{3\alpha-1} (\zeta_0-\zeta)^{(3\alpha-1)/(\alpha-1)} \lambda^{2\alpha/(\alpha-1)}, \tag{A 8}$$

and so the neglect of $f_{\eta\eta\eta}$ in (A 5) is no longer justified. It may be shown that this term is important in a region near $\zeta = \zeta_0$ of width $T^{\frac{1}{4}}$, and accordingly we write

$$\eta = \zeta_0 T^{-1/2\alpha} + \chi T^{\frac{1}{4}}, \tag{A 9}$$

$$f = -\frac{\zeta_0 T^{-1/2\alpha}}{\tau_s(1-m)} + \eta_1 - \frac{\chi T^{\frac{1}{4}}}{\tau_s(1-m)} + \lambda \chi T^{\frac{3}{4}} + T^{\mu} H(\chi), \tag{A 10}$$

where H is a function of χ to be found and

$$\mu = \frac{5\alpha+1}{4(\alpha-1)}. \tag{A 11}$$

For (A 10) to be a solution of (2.5)

$$H''' + \tfrac{1}{2}\tau_s(1-m)\,\lambda[H'(2\mu+\tfrac{1}{2}) - \tfrac{3}{2}\chi H''] = 0, \tag{A 12}$$

and to match with (A 8)

$$H \simeq -\tfrac{1}{2}B\lambda^{(\alpha+1)/(\alpha-1)}(-\chi)^{(3\alpha-1)/(\alpha-1)} \quad \text{as} \quad \chi \to -\infty. \tag{A 13}$$

A solution of (A 12) that satisfies (A 13) can always be found, but in general it becomes exponentially large as $\chi \to \infty$ and must be rejected. The only exceptions are when

$$4\mu+1 = 3, \quad \alpha = \frac{n}{n-2}, \tag{A 14}$$

where $n \geqslant 2$ is an integer. This condition fixes the allowed values of α and (A 6) now reduces to

$$g'^2 = \lambda^2 + B(-g)^n. \tag{A 15}$$

The simplest case $n = 2$ corresponds to Sychev's solution; if we wish f_η to have a minimum then n must be even. Let us fix attention on this possibility now so that we are considering alternatives to the description of §3, and $n = 2p$, where p is an integer. Then g passes smoothly from negative to positive values at $\zeta = \zeta_0$, where

$$\zeta_0 = \frac{2}{\pi^{\frac{1}{2}}}(n-2)\left(\frac{\lambda}{A}\right)^{2/n-1}\left(\frac{1}{n}\right)!\left(-\frac{1}{n}-\frac{1}{2}\right)!, \tag{A 16}$$

and subsequently g' increases from the value λ. This solution comes to an end at $\zeta = 2\zeta_0$, when both g and g' become infinite. In particular,

$$g' \sim pA(2p)^{-1/2(p-1)}(2\zeta_0-\zeta)^{-p/(p-1)} \tag{A 17}$$

as $\zeta \to 2\zeta_0 -$. We may now revert back to the original variable η, apart from an origin shift, writing

$$\eta = 2\zeta_0 T^{-1/2\alpha}+\theta, \tag{A 18}$$

and then

$$f_\eta \sim -\frac{1}{\tau_s(1-m)}+A(-\theta)^{-\alpha},$$

where θ is large and negative. In a similar way

$$f+\frac{2\zeta_0 T^{1/2\alpha}}{\tau_s(1-m)} \tag{A 19}$$

is also a function of θ only when θ is large and negative and $T\to 0$. Thus when θ is $O(1)$, f_η is a function of θ only, which approaches the limit $+1$ as $\theta\to\infty$. The outer part of the boundary layer when the skin friction is positive is pushed away from the wall but otherwise remains finite in extent. The displacement thickness δ^* satisfies

$$\delta^* \sim \frac{2\zeta_0(\tau_s-\tau)^{-1/2\alpha}}{\tau_s(1-m)}(1+\tau_s(1-m)). \tag{A 20}$$

If n is odd, the structure proposed in this appendix has a possible application to singularities associated with the vanishing of $1+(1-m)\tau f_\eta$ even though f_η is monotonic (decreasing). Generally this term might vanish at a finite value η_2 of η, and as the singularity is approached it is necessary that $\eta_2\to\infty$. In addition (A 1) must hold when $T \ll 1$ and η is large but finite.

Then formally a parallel match may be made, for the leading terms in the expansion about $T = 0$, to that when n is even. The solution, for ζ finite, automatically matches with (A 1) as $\zeta\to 0$, and, as ζ increases, g is initially negative while $g' > 0$. At $\zeta = \zeta_0$, $g' = \lambda$ and g vanishes. At $\zeta = \zeta_1$, where

$$\zeta_1 = \int_{-\infty}^{g_1}\frac{dg}{[\lambda^2+B(-g)^n]^{\frac{1}{2}}} = \frac{2}{\pi^{\frac{1}{2}}}\frac{\lambda^{2/n-1}}{B^{1/n}}\left(\frac{1}{n}\right)!\left(-\frac{1}{n}-\frac{1}{2}\right)!\cos^2\frac{\pi}{2n}, \tag{A 21}$$

g reaches a maximum value of $g_1 = (\lambda^2/B)^{-1/n}$. Thereafter, g' becomes negative and both g and $g'\to-\infty$ as $\zeta\to 2\zeta_1$. Thus the limit profile when ζ is finite is monotonic but it has two points of inflexion on either side of $g' = 0$. In terms of η, these points are all large distances apart. In the neighbourhood of $2\zeta_1$, we may revert to the original variable η, again apart from an origin shift, and write

$$\eta = 2\zeta_1 T^{-1/2\alpha}+\phi. \tag{A 22}$$

Then

$$f_\eta \sim -\frac{1}{\tau_s(1-m)}-A(-\phi)^{-\alpha} \tag{A 23}$$

when ϕ is large and negative, confirming that χ changes sign when T is small. Should the outer part of the boundary layer be $O(1)$ in thickness, then the overall displacement thickness δ^* also satisfies (A 20) with ζ_0 replaced by ζ_1. The weakest singularity in δ^* occurs when $n = 3$, and then

$$\delta^* \propto (\tau_s-\tau)^{-\frac{1}{6}} \tag{A 24}$$

as $\tau \to \tau_s$. If ζ is finite, the determination of further terms in the expansion of f in powers of T (cf. 3.10) requires the solution of a sequence of linear ordinary differential equations with or without forcing terms and having g as coefficient of the highest derivative. It is possible that non-analytic solutions may be generated at any zero of g, of which there is one when n is even and two when n is odd. Further study is required to determine the consequences for the structure proposed here. It may even be that a contradiction is eventually reached, nullifying the postulate of structure, but this seems unlikely, for $n = 2$ at least, in view of the good agreement with the numerical studies. We also note that in the theory of singularities of unsteady boundary layers on rotating disks (Stewartson, Simpson & Bodonyi 1982) the situation for the higher terms in the expansion is somewhat similar to that here for n odd.

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